A Bayesian Analysis in Determining Pro-rata Warranty Decision with Two-Dimensional Deterioration

Po-Chiang Tsai and Chih-Chiang Fang

Abstract—In practice, the condition of warranty should be taken into consideration for customers in order to get better post-sale service. Accordingly, manufacturers would use this as a marketing tool to advertise the quality of their products. However, offering warranty usually results in additional costs to the manufacturers. Therefore, the manufacturers would balance the costs and the benefits when they make the related decisions. Moreover, the deterioration of a product not only depends on time but also on usage. Under such situation, only considering one of them could misrepresent the estimation of products’ deterioration, and therefore a two-dimensional failure model would be fit for dealing with such problems. Furthermore, most of the studies regarding pro-rata warranty issue mainly focused on cost estimation or reliability estimation but few studies would consider how the marketing strategies can be integrated into a synthetic decision. Another important issue is the manufacturer may not have sufficient historical data to estimate the deterioration of a newly developed product, and therefore the results obtained from analytical models may not be reliable. In order to deal with such a problem for the situation of insufficient historical data, a Bayesian analysis should be a reasonable approach. Accordingly, this paper proposes a Bayesian decision model which considers the pro-rata warranty with the pricing and the production strategy for the manufacturers.

Index Terms—Pro-rata warranty, repairable products, Bayesian analysis.

I. INTRODUCTION

A product warranty is a contract offered by a manufacturer to its customers to repair or replace faulty items. From the buyer’s perspective, the main role of a warranty is to safeguard the transaction. On the other hand, from the producer’s perspective is a promotional instrument, similar to the price and performance of a product, to be used in competing with other manufacturers in the marketplace. However, it should be noted that offering unlimited warranty, with the intent of monopolizing the market, is unrealistic, since the costs will eventually outstrip the profits. Therefore, an appropriate warranty is important to the manufacturer.

Generally speaking, there are two basic warranty policies provided by the manufacturers. They are the free replacement warranty (FRW), which charges consumers no fee during the term of warranty, and the pro-rata warranty (PRW), which charges the consumer a preset proportion of cost for each repair within the term of warranty. From a practical point view, the PRW is a popular warranty policy for relatively high-priced products, such as plant facilities and large scale machines. However, the more the warranty proportion and length, the higher the operational costs will get, and which will eventually turn the product into unprofitable.

Most of studies regarding PRW issue mainly focus on the cost analysis with the consideration of reliability. The studies include the following: Menke [1] evaluated the warranty cost for a non-repairable product under PRW by the assumption of an exponential failure process. Balcer and Sahin [2] proposed a stochastic failure distribution for deriving the expected warranty cost in which the repair cost is paid by the manufacturer in accordance with the proportion of damage under PRW. Blischke and Murthy [3] derived the expected warranty cost and the profit for renewable and non-renewable PRW from the perspectives of sellers and buyers, respectively under the assumption of Weibull failure model. Wang et al. [4] stated that warranty cost is positively related to products’ reliability. By improving the reliability of products, the expected warranty cost can be reduced. Jain and Maheshwari [5] propose a warranty model for the renewing PRW, in which the failure rate of units, cost of preventive maintenance and cost of replacement are assumed to be constant. The proposed model can determine the optimal number and optimal period for preventive maintenance after the expiry of the warranty. Wu et al. [6] developed a cost model to determine the optimal burn-in time and warranty length for non-repairable products under FRW/PRW policy. Based on the results of the analysis, the fully renewing combination FRW/PRW is always better than the fully renewing policy in terms of cost.

In some cases, the manufacturer does not have sufficient historical data to estimate the deterioration of a newly developed product, the results obtained from analytical models may not be reliable. In such situations, the Bayesian analysis is a reasonable approach to additionally take expert opinions into account for better decision making. Papazoglou [7] used a Bayesian decision analysis to solve the problem of reliability certification. Papazoglou’s approach is based on the existing prior assessment of uncertainties and further information can be obtained by testing the components. Juang and Anderson [8] utilized a Bayesian approach to determine an optimal adaptive preventive maintenance policy with minimal repairs.

Due to the fact that the deterioration of a product depends not only on time but also on usage. Under such situation, only considering one of them could misrepresent the estimation of equipment’s deterioration. Therefore, two-dimensional failure model would be fit for dealing with such problems. Huang and Yen [9] provided manufacturers the guidelines.
that how to offer customers two-dimensional warranty programs with proper time and usage limits. Shahanaghi et al. [10] designed a two-dimensional extended warranty contract for automobile-manufacturers.

In this paper, we therefore proposed a mathematical model to help the manufacturer in systematically deciding pricing, production and warranty policy under insufficient information. The failure times of the product were assumed to be drawn from a non-homogeneous Poisson process (NHPP) with a specific intensity function. In the next section, we state and formulate the problem of the proposed model.

II. ONE-DIMENSIONAL WARRANTY MODEL

In this paper, the more general case of non-renewable products is focused, and the warranty policy under considerations is PRW. It is of special interest under the policy of PRW that the warranty proportion and term is a difficult trade-off, since the less the warranty proportion and the shorter warranty term are, the more cutback will be in the warranty cost, on the contrary, the more the warranty proportion and the longer warranty term are, the more attractive the product could be, and which may result in more marketing sales. In such case, the number of breakdowns or failures during the warranty term becomes crucial in choosing the optimal proportion for PRW because each breakdown will increase the warranty cost due to the required repairs and replacements. In line with the prior work, we assume that the nonhomogeneous Poisson process (NHPP) can properly describe the breakdown process of the product based on the following assumptions: 1) the repaired product is as good as old (i.e., minimal repair); 2) the repair time is negligible (Ascher and Feingold [11]), and the intensity function of the deterioration NHPP is \( \lambda(t) \) which characterizes the deteriorating behavior of the process. Huang and Bier [12] suggested that the power law intensity function of the form \( \lambda(t)=\alpha \beta t^{-1} \) is more flexible and manageable than other used model, such as linear, exponential, or other mixed type intensity functions to appropriately describe the deteriorating behavior of deteriorating systems. Besides, Huang and Bier proposed a natural conjugate prior distribution for a power law deteriorating model which can be facilitated to analytically discuss the aforementioned decision problem, whereas other approaches with more general forms of intensity function may not be capable of performing the similar analysis. Therefore, in this paper, we simply assume that the successive failure times of the repairable product is drawn from an NHPP with a power law intensity function.

Since the synthetic warranty decisions would involve both the revenue of sales and the cost of production, the manufacturer has to realize these issues and take them into account. The demand function extended by Glickman and manufacturer has to realize these issues and take them into account. The demand function extended by Glickman and manufacturer has to realize these issues and take them into account. The demand function extended by Glickman and

\[
N_{\varphi}(\alpha, \beta, W, h) = \sum_{k=1}^{\lfloor W/h \rfloor - 1} \int_{(k-1)h}^{kh} \lambda(t) dt + \int_{[W/h]}^{W} \lambda(t) \frac{W - W/h}{h} dt
\]

where \( \lfloor W/h \rfloor \) is the lower integer part of the number of times of maintenance within the warranty period \([0, W]\). \( \sum_{k=1}^{\lfloor W/h \rfloor - 1} \int_{(k-1)h}^{kh} \lambda(t) dt \) is the expected number of failures within the time period \([0, W/h]\), and \( \int_{[W/h]}^{W} \lambda(t) \frac{W - W/h}{h} dt \) is the surplus of the expected number of failures after the \([W/h]\)th maintenance.

Due to mechanical aging of the deteriorating system, the each maintenance cost will get higher and higher. Accordingly, the maintenance cost per unit during the warranty period is (according to the assumptions of Jayabal and Chaudhuri [14])

\[
C_M(C_F, \tau, h, W) = \sum_{k=1}^{\lfloor W/h \rfloor} C_F(1 + \tau kh)
\]

where \( C_F \) denotes the basic value of each preventive maintenance, and \( \tau \) denotes the annual increasing rate of the preventive maintenance cost. Owing to that the PRW policy is adopted, the repair cost of the manufacturer’s duty should be

\[
\rho E[C_F]N_{\varphi}(\alpha, \beta, W, h, \eta).
\]

From all the above discussions, the expected profit can be formulated as follows:
The usage rate $s$ is a ratio of the using time to the usage for the usage rates. In other words, it denotes the usage per unit time of a product. Due to the fact that the usage rate $s$ is very helpful to transform the two-variate problem into univariate problem, we can utilize this to re-estimate the expected failures of products in consideration of all customers’ usual practice. The Fig. 1 illustrates the relation between the using time and the usage for the usage rates.

Fig. 1. The relation between the using time and the usage for the usage rates.

Suppose that the usage rate $s$ is a random variable and approximate to Gamma distribution according to all customers’ usual practice, and therefore the estimation of the expected number of breakdowns should be rewritten as follow:

$$\int_{0}^{T} \int_{0}^{U_s}(t,s) \times \text{Gamma}(s|\theta,\gamma)dsdt$$

where $\text{Gamma}(s|\theta,\gamma)$ is Gamma probability density function under the parameters $\theta$ and $\gamma$, and its form can be expressed as

$$\text{Gamma}(s|\theta,\gamma) = \frac{s^{\theta-1}e^{-s\gamma}}{\gamma^\theta \Gamma(\theta)}$$

Therefore, with consideration of preventive maintenance program, the expected number of minimal repairs with the age reduction factor $\eta$ during the period $[0, W]$ under the time interval $h$ between maintenance visits should be

$$\sum_{k=1}^{h} \int_{t-h}^{t} \left[ \frac{\alpha_i \beta_i (t-h)^{\beta_i-1}}{\alpha \beta \eta \Gamma(\theta)} \right] \times \text{Gamma}(s|\theta,\gamma)dsdt$$

From the above discussion, the expected profit with consideration of two-dimensional warranty can be formulated as follows:

$$\text{Max} \{ \pi(W_i, \rho_j, P, Q) \} = P Q - C_p Q$$

$$- \rho_j E[C_{N_p} | N_R N_R(\alpha, \beta, W, h, \eta)]Q$$

$$- C_M (C_F \tau, h, W_t)Q$$

Subject to: $Q = \omega_i P^{\psi_i} (\omega_2 + \rho_2^\psi_i W_i)^{\psi_2}$

The following assumptions should be noticed before you
employ the model of the study:
1) The products’ deteriorates due to time and usage, and the failure or breakdown process can be described by an NHP.
2) A minimal repair is performed when a failure or breakdown occurs within the warranty period. After the repair, the product would be restored to its former status immediately.
3) The PM activities are imperfect so the condition of the product cannot fully restore the previous status after PM.
4) The probability distribution of the usage rate of the products is assumed to Gamma distribution.

IV. BAYESIAN ANALYSIS

Suppose an NHPP with the power law intensity function can appropriately model the breakdown process. The optimal warranty proportion and term will be selected in accordance with the knowledge of the unknown parameters, which will be treated as random variables for Bayesian analysis, along with other considerations about costs from multiple perspectives.

Before proceeding with Bayesian analysis, the decision makers should verify that the assumptions are satisfied. The domain experts can provide the value of parameters $u_a$, $u_o$, and $\sigma_{\beta}$ according to prior knowledge. Besides, the market department would need to estimate the demand function of the new product based on market survey and analysis. After the above preparations, the decision makers can proceed with the prior analysis for the problem, and make the decision as long as they convinces the results of the prior analysis. If the decision makers consider that the prior analysis may not be very convincing and acceptable, the posterior analysis can be effortlessly gained without further computation by the property of natural conjugate family, and which is given by

\[ f \left( \alpha, \beta \right) \propto L \left( D^{(n)} | \alpha, \beta \right) f \left( \alpha, \beta \right) = K' \alpha^{m+n-1} \beta^{m+n-1} \left( e^{-d z^m} \prod_{i=1}^{n} x_i \right)^{\beta-1} e^{-n(cz^d + x_i^d)} \]  

(14)

However, the expectation of failures for the posterior analysis cannot be obtained in close form expression because the posterior distribution in Equation (14) is much more complex than the prior distribution in Equation (12). Fortunately, numerical integration can be employed to calculate the expectation of failures for the posterior analysis, which is given by

\[ E \left[ \alpha T^\beta \right] = \int_A \int_B \alpha T^\beta f \left( \alpha, \beta \right) d\beta d\alpha \]

(15)

The posterior analysis can therefore be carried out with the considerations of both prior knowledge and sampling information.

V. CONCLUSION

The objective of this paper is to provide a methodology to solve the synthesized decision which concerns both warranty and marketing perspectives under the situation of scarce historical data. In the past, few studies discussed how the warranty proportion influences the related costs and profits so that the manufacturer might not take the advantage of the
information to make a better decision. The future work could also be performed on refining the proposed model by considering the repair cost as a function of age, since the repair cost is subject to the degree of mechanical aging. A compound NHPP might be suitable for this analysis. It would be interesting to consider that the warranty proportion is subject to change as time elapses. However, such analysis would be expected to be more complicated to perform.

ACKNOWLEDGMENT

This work was supported by the Ministry of Science and Technology in Taiwan, under Chih-Chiang Fang’s Grant: MOST-103-2410-H-366-002.

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